TOPIC 5
APPLICATIONS OF DERIVATIVES
SCHEMATIC DIAGRAM

Topic	Concepts	Degree of importance	Refrences NCERT Text Book XII Ed. 2007
Application of Derivative.	1.Rate of change	*	Example 5 Ex 6.1 Q.No- 9,11
	2.Increasing & decreasing functions	***	Ex 6.2 Q.No- 6 Example 12,13
	3.Tangents & normals	**	Ex 6.3 Q.No- 5,8,13,15,23
	4.Approximations	*	Ex 6.4 QNo- 1,3
	5 Maxima & Minima	***	Ex 6.5Q.No- 8,22,23,25 Example 35,36,37,

SOME IMPORTANT RESULTS/CONCEPTS

** Whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left[\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

** Let I be an open interval contained in the domain of a real valued function f. Then f is said to be
(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in$ I.
(ii) strictly increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.
(iii) decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \ge f(x_2)$ for all $x_1, x_2 \in$ I.
(iv) strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in$ I.

** (i) f is strictly increasing in (a, b) if $f'(x) > 0$ for each $x \in (a, b)$
(ii) f is strictly decreasing in (a, b) if $f'(x) < 0$ for each $x \in (a, b)$
(iii) A function will be increasing (decreasing) in **R** if it is so in every interval of **R**.

** Slope of the tangent to the curve $y = f(x)$ at the point (x_0, y_0) is given by $\left[\frac{dy}{dx}\right]_{(x_0, y_0)} \left(= f'(x_0)\right)$.

** The equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by $y - y_0 = f'(x_0)(x - x_0)$.

** Slope of the normal to the curve $y = f(x)$ at (x_0, y_0) is $-\frac{1}{f'(x_0)}$.

** The equation of the normal at (x_0, y_0) to the curve $y = f(x)$ is given by $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$.

** If slope of the tangent line is zero, then $\tan\theta = 0$ and so $\theta = 0$ which means the tangent line is parallel to the

x-axis. In this case, the equation of the tangent at the point (x0, y0) is given by $y = y_0$.

** If $\theta \to \frac{\pi}{2}$, then $\tan\theta \to \infty$, which means the tangent line is perpendicular to the x-axis, i.e., parallel to the y-axis. In this case, the equation of the tangent at (x_0, y_0) is given by $x = x_0$.

** Increment Δy in the function $y = f(x)$ corresponding to increment Δx in x is given by $\Delta y = \frac{dy}{dx}\Delta x$.

** Relative error in $y = \frac{\Delta y}{y}$.

** Percentage error in $y = \frac{\Delta y}{y} \times 100$.

** Let f be a function defined on an interval I. Then
(a) f is said to have a maximum value in I, if there exists a point c in I such that $f(c) \ge f(x)$, for all $x \in$ I.
The number f (c) is called the maximum value of f in I and the point c is called a point of maximum value of f in I.
(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) \le f(x)$, for all $x \in$ I.
The number f (c), in this case, is called the minimum value of f in I and the point c, in this case, is called a point of minimum value of f in I.
(c) f is said to have an extreme value in I if there exists a point c in I such that f (c) is either a maximum value or a minimum value of f in I.
The number f (c), in this case, is called an extreme value of f in I and the point c is called an extreme point.

* * Absolute maxima and minima
Let f be a function defined on the interval I and $c \in$ I. Then
(a) f(c) is absolute minimum if f(x)□$\ge$ f(c) for all $x \in$ I.
(b) f(c) is absolute maximum if $f(x) \le f(c)$ for all $x \in$ I.
(c) $c \in$ I is called the critical point off if $f'(c) = 0$
(d) Absolute maximum or minimum value of a continuous function f on [a, b] occurs at a or b or at critical points off (i.e. at the points where f ′is zero)
If $c_1, c_2, \dots, c_n$ are the critical points lying in [a , b], then
absolute maximum value of $f = \max\{f(a), f(c_1), f(c_2), \dots, f(c_n), f(b)\}$
and absolute minimum value of $f = \min\{f(a), f(c_1), f(c_2), \dots, f(c_n), f(b)\}$.

** Local maxima and minima
(a)A function f is said to have a local maxima or simply a maximum vaJue at x a if $f(a \pm h) \le f(a)$ for sufficiently small h
(b)A function f is said to have a local minima or simply a minimum value at x = a if $f(a \pm h) \ge f(a)$.

** First derivative test : A function f has a maximum at a point x = a if
(i) $f'(a) = 0$, and
(ii) $f'(x)$ changes sign from + ve to –ve in the neighbourhood of 'a' (points taken from left to right).
However, f has a minimum at x = a, if
(i) $f'(a) = 0$, and
(ii) $f'(x)$ changes sign from –ve to +ve in the neighbourhood of 'a'.
If $f'(a) = 0$ and $f'(x)$ does not change sign, then f(x) has neither maximum nor minimum and the point 'a' is called point of inflation.
The points where $f'(x) = 0$ are called stationary or critical points. The stationary points at which the function attains either maximum or minimum values are called extreme points.

** Second derivative test

(i) a function has a maxima at x a if $f'(x)$ 0 and $f''(a) < 0$
(ii) a function has a minima at x = a if $f'(x) = 0$ and $f''(a) > 0$.

ASSIGNMENTS

1.Rate of change

LEVEL -I

1. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.
2 .The side of a square sheet is increasing at the rate of 4 cm per minute. At what rate is the area increasing when the side is 8 cm long ?
3. The radius of a circle is increasing at the rate of 0.7 cm/sec. what is the rate of increase of its circumference ?

LEVEL –II

1. Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate?
2. A man 2 metre high walks at a uniform speed of 6km /h away from a lamp post 6 metre high. Find the rate at which the length of his shadow increases. Also find the rate at which the tip of the shadow is moving away from the lamp post.
3. The length of a rectangle is increasing at the rate of 3.5 cm/sec and its breadth is decreasing at the rate of 3cm/sec. find the rate of change of the area of the rectangle when length is 12 cm and breadth is 8 cm

LEVEL III

1. A particle moves along the curve $6y = x^3 + 2$., Find the points on the curve at which y-coordinate is changing 8 times as fast as the x-coordinate.
2. Water is leaking from a conical funnel at the rate of 5 cm^3/sec. If the radius of the base of the funnel is 10 cm and altitude is 20 cm, Find the rate at which water level is dropping when it is 5 cm from top.
3. From a cylinder drum containing petrol and kept vertical, the petrol is leaking at the rate of 10 ml/sec. If the radius of the drum is 10cm and height 50cm, find the rate at which the level of the petrol is changing when petrol level is 20 cm

2.Increasing & decreasing functions

LEVEL I

1. Show that $f(x) = x^3 - 6x^2 + 18x + 5$ is an increasing function for all $x \in R$.
2. Show that the function $x^2 - x + 1$ is neither increasing nor decreasing on (0,1)
3. Find the intervals in which the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$ isincreasing or

decreasing.

LEVEL II

1. Indicate the interval in which the function $f(x) = \cos x$, $0 \le x \le 2\pi$ is decreasing.

2.Show that the function $f(x) = \frac{\sin x}{x}$ is strictly decreasing on $(0, \pi/2)$

3. Find the intervals in which the function $f(x) = \frac{\log x}{x}$ increasing or decreasing.

LEVEL III

1. Find the interval of monotonocity of the function $f(x) = 2x^2 - \log x$, $x \neq 0$
2. Prove that the function $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ in $[0, \pi/2]$

[CBSE 2011]

3.Tangents &Normals

LEVEL-I

1.Find the equations of the normals to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.
2. Find the point on the curve $y = x^2$ where the slope of the tangent is equal to the x-coordinate of the point.

3. At what points on the circle $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangent is parallel to x axis ?

LEVEL-II

1. Find the equation of the normal to the curve $ay^2 = x^3$ at the point (am^2, am^3)

2. For the curve $y = 2x^2 + 3x + 18$, find all the points at which the tangent passes through the origin.

3. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$
4. Show that the equation of tangent at (x_1, y_1) to the parabola $yy_1 = 2a(x + x_1)$. **[CBSE 2012Comptt.]**

LEVEL- III

1 .Find the equation of the tangent line to the curve $y = \sqrt{5x-3} - 2$ which is parallel to the line $4x - 2y + 3 = 0$

2. Show that the curve $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 - 2y = 0$ cut orthogonally at the point (0,0)

3. Find the condition for the curves $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $xy = c^2$ to intersect orthogonally.

4.Approximations

LEVEL-I

Q.1 Evaluate $\sqrt{25.3}$

Q.2 Use differentials to approximate the cube root of 66

Q.3 Evaluate $\sqrt{0.082}$

Q.4 Evaluate $\sqrt{49.5}$ **[CBSE 2012]**

LEVEL-II

1. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area

5 Maxima & Minima

LEVEL I

1. Find the maximum and minimum value of the function $f(x) = 3 - 2 \sin x$

2. Show that the function $f(x) = x^3 + x^2 + x + 1$ has neither a maximum value nor a minimum value

3. Find two positive numbers whose sum is 24 and whose product is maximum

LEVEL II

1. Prove that the area of a right-angled triangle of given hypotenuse is maximum when the triangle is isosceles.

2.A piece of wire 28(units) long is cut into two pieces. One piece is bent into the shape of a circle and other into the shape of a square. How should the wire be cut so that the combined area of the two figures is as small as possible.

3. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

LEVEL III

1 .Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse having its vertex coincident with one extremity of major axis.

2.An open box with a square base is to be made out of a given quantity of card board of area c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.**[CBSE 2012 Comptt.]**

3.A window is in the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window. **[CBSE 2011]**

Questions for self evaluation

1.Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

2. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?

3. Find the intervals in which the following function is strictly increasing or decreasing:
$f(x) = -2x^3 - 9x^2 - 12x + 1$

4. Find the intervals in which the following function is strictly increasing or decreasing:
$f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$

5. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

6. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is
(a) parallel to the line $2x - y + 9 = 0$ (b) perpendicular to the line $5y - 15x = 13$.

7. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

8. Using differentials, find the approximate value of each of the following up to 3places of decimal :

(i) $(26)^{\frac{1}{3}}$ (ii) $(32.15)^{\frac{1}{5}}$

9. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

10. An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of aluminium and folding up the sides. Find the volume of the largest such box.

Answers

TOPIC 5 APPLICATIONS OF DERIVATIVES

1. Rate of change

LEVEL I 1. $\frac{27\pi}{8}(2x+1)^2$ 2. 64 cm^2/min 3. 4.4 cm/sec

LEVEL II 1. (2 , 4) 2. 9 km/h 3. 8 cm^2/sec

LEVEL III 1. (4 , 11) and $\left(-4, \frac{-31}{3}\right)$ 2. $\frac{4}{45}\pi$cm/sec 3. $\frac{1}{10}\pi$cm/sec

2. Increasing & decreasing functions

LEVEL I 3. (0, 3π/4) U (7π/4, 2π) and (3π/4, 7π/4)

LEVEL II 1. (0 ,π) 3. (0 , e) and (e, ∞)

LEVEL III 1. (- ½ , 0) U (½ , ∞) &(-∞ , -1/2) U (0,1/2)

3. Tangents &normals

LEVEL I 1. x + 3y - 8 = 0 & x + 3y + 8 =0 2. (0 , 0)
3. (1 , 0) & (1 , 4)

LEVEL II 1. $2x + 3my - 3am^4 - 2am^2 = 0$ 2. (3 , 45) & (–3 , 27)
3. x + 14y – 254 = 0 & x + 14y + 86 = 0

LEVEL III 1. 80x – 40y – 103 = 0 3. $a^2 = b^2$[Hint: Use $m_1 m_2 = -1$]

4. Approximations

LEVEL I 1. 5.03 2. 4.042 3. 0.2867 4. 7.036

LEVEL II 1. 2.16 π cm

5 Maxima & Minima

LEVEL I 1. 1 & 5 3. 12, 12

LEVEL II 2. $\frac{112}{\pi+4}$ cm , $\frac{28\pi}{\pi+4}$ cm. 3. Length = $\frac{20}{\pi+4}$ m , bredth $\frac{10}{\pi+4}$ m.

LEVEL III 1. $\frac{3\sqrt{3}}{4}$ab 3. $\frac{4(6+\sqrt{3})}{11}$m , $\frac{30-6\sqrt{3}}{11}$

Questions for self evaluation

1. $\frac{1}{48\pi}$cm/s 2. $b\sqrt{3}cm^2/s$ 3. $\uparrow$ in $(-2, -1)$ and $\downarrow$ in $(-\infty, -2) \cup (-1, \infty)$

4. $\uparrow$ in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$ and $\downarrow$ in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ 5. (0 , 0) , (1 , 2), (– 1, – 2)

6. (a) y – 2x – 3 = 0 , (b) 36y + 12x – 227 = 0 8.(i) 2.962 (ii) 2.962

10. $\frac{200}{7}$ m